

Detection in the Presence of Additive Noise and Unknown Offset

Kees A. Schouhamer Immink and Jos H. Weber

Abstract - *The error performance of optical storage and Non-Volatile Memory (Flash) is susceptible to unknown offset of the retrieved signal. Balanced codes offer immunity against unknown offset at the cost of a significant code redundancy, while minimum Pearson distance detection offers immunity with low-redundant codes at the price of lessened noise margin. We will present a hybrid detection method, where the distance measure is a weighted sum of the Euclidean and Pearson distance, so that the system designer may trade noise margin versus amount of immunity to unknown offset.*

Key words: Constant composition code, permutation code, flash memory, digital data storage, Non-Volatile Memory, NVM, mismatch, fading, Pearson distance, Euclidean distance, parity check.

I. INTRODUCTION

The receiver of a transmission or storage system is often ignorant of the exact value of the offset, also known as drift or baseline wander, of the received signal. In optical data storage, both the amplitude and offset of the retrieved signal depend on the reflective index of the disc surface and the dimensions of the written features [1]. In non-volatile memories, such as floating gate memories, the data is represented by stored charge, which can leak away. The amount of leakage depends on various physical parameters and, clearly, on the time elapsed between writing and reading the data. In such a case, the conventional approach of combatting the effects of channel mismatch by using an automatic offset or gain control that estimates the mismatch from the previously received data can not be applied, and other methods have to be developed.

In optical disc storage devices and non-volatile memories, constrained codes, specifically *dc-free* or *balanced* codes, have been used and/or proposed to counter the effects of offset mismatch [2]. Jiang *et al.* [3] addressed a q -ary balanced coding technique, called *rank modulation*, for circumventing the difficulties with flash memories having aging offset levels. Zhou *et al.* [4], Sala *et al.* [5], and Immink [6] investigated the usage of balanced codes for

enabling ‘dynamic’ reading thresholds in non-volatile memories. Alternative detection methods, such as minimum Pearson distance detection, that are immune to offset and gain mismatch, have been presented in [7]. Minimum Pearson distance detection offers immunity to both gain and offset mismatch, which is bought at the price of a lessened noise margin.

We will investigate the properties of a hybrid detector that weighs minimum Pearson and minimum Euclidean distance measures, where the designer has the option of trading the amount of mismatch immunity on the one hand versus loss of noise margin on the other hand. We start, in Section II, with a description of hybrid minimum Pearson and Euclidean distance detection. In Section III, we present an analysis of the error performance of the hybrid detector, and a mismatch sensitivity analysis. In Section IV, we will show that a simple rate $1 - 1/n$ parity-check code may improve the noise margin by slightly less than 3 dB. In Section V, we will discuss implementation issues, and in Section VI, we will describe our conclusions.

II. DESCRIPTION OF HYBRID DETECTION

We consider a communication codebook, S , of chosen q -ary codewords $\mathbf{x} = (x_1, x_2, \dots, x_n)$ over the q -ary alphabet $\mathcal{Q} = \{0, 1, \dots, q-1\}$, $q \geq 2$, where n , the *length* of \mathbf{x} , is a positive integer. Suppose the sent codeword, \mathbf{x} , is received as the vector $\mathbf{r} = \mathbf{x} + \boldsymbol{\nu} + b$, $r_i \in \mathbb{R}$, where $b \in \mathbb{R}$ is an unknown (dc-)offset, and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$, $\nu_i \in \mathbb{R}$ are noise samples with distribution $N(0, \sigma^2)$. We use the shorthand notation $\mathbf{x} + b = (x_1 + b, x_2 + b, \dots, x_n + b)$.

Before we proceed with a description of the new detector, we will define T -constrained codes. T -constrained codes, presented in [6] for enabling simple dynamic threshold detection of q -ary codewords, consist of q -ary n -length codewords, where T , $0 < T \leq q$, *preferred* or *reference* symbols must appear at least once in a codeword. The code, denoted by S_1 , comprises codewords where the symbol ‘0’ appears at least once. The size of S_1 equals [6]

$$|S_1| = q^n - (q-1)^n, q > 1. \quad (1)$$

For the binary case, $q = 2$, we simply find

$$|S_1| = 2^n - 1,$$

the all-‘1’ word is deleted. We will assume that the code used is $S = S_1$.

Immink & Weber [7] advocated the usage of the Pearson distance for channels with both gain and offset mismatch.

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For channels with offset mismatch only, they introduced the modified Pearson distance

$$\delta_p(\mathbf{r}, \hat{\mathbf{x}}) = \sum_{i=1}^n (r_i - \hat{x}_i + \bar{x})^2, \quad (2)$$

between \mathbf{r} and $\hat{\mathbf{x}}$, where \mathbf{x} are $\hat{\mathbf{x}}$ are taken from S_1 , and

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n \hat{x}_i. \quad (3)$$

The receiver outputs the codeword that is closest to \mathbf{r} , or

$$\mathbf{x}_o = \arg \min_{\hat{\mathbf{x}} \in S_1} \delta_p(\mathbf{r}, \hat{\mathbf{x}}). \quad (4)$$

It has been shown in [7] that the outcome (4) is intrinsically resistant to the unknown offset, b , at the cost of noise margin.

We define a new hybrid distance measure by

$$\delta(\mathbf{r}, \hat{\mathbf{x}}) = \gamma \delta_e(\mathbf{r}, \hat{\mathbf{x}}) + (1 - \gamma) \delta_p(\mathbf{r}, \hat{\mathbf{x}}), \quad (5)$$

where

$$\delta_e(\mathbf{r}, \hat{\mathbf{x}}) = \sum_{i=1}^n (r_i - \hat{x}_i)^2 \quad (6)$$

is the (squared) Euclidean distance between the received signal vector \mathbf{r} and the codeword $\hat{\mathbf{x}} \in S_1$. The parameter γ , $0 \leq \gamma \leq 1$, can be judiciously chosen for trading the sensitivity to offset mismatch versus noise margin loss. Clearly, for $\gamma = 1$ we obtain a pure minimum Euclidean distance detector, which offers optimal noise margin, but it is sensitive to channel mismatch. On the other hand, for $\gamma = 0$ we obtain a pure minimum Pearson distance detector, which is immune against offset mismatch, but it comes at a lessened noise margin. Other values of γ will lead to a detector that balances the properties of the two types of detectors. The hybrid detector outputs the codeword

$$\mathbf{x}_o = \arg \min_{\hat{\mathbf{x}} \in S_1} \delta(\mathbf{r}, \hat{\mathbf{x}}). \quad (7)$$

A direct implementation of (7) requires around q^n computations of (5), which is prohibitive for larger values of n . In Section V, we will show that the computation of (7) can be accomplished with much less computational load, for example, for $q = 2$, only n instead of $2^n - 1$ evaluations of (5) are needed.

In the next section, we will compute the error performance of the new detector.

III. ERROR PERFORMANCE ANALYSIS

The detector errs if there is at least one codeword $\hat{\mathbf{x}} \neq \mathbf{x}$, $\hat{\mathbf{x}} \in S_1$, such that

$$\delta(\mathbf{r}, \hat{\mathbf{x}}) < \delta(\mathbf{r}, \mathbf{x}) \quad (8)$$

or

$$\gamma(\delta_e(\mathbf{r}, \hat{\mathbf{x}}) - \delta_e(\mathbf{r}, \mathbf{x})) + (1 - \gamma)(\delta_p(\mathbf{r}, \hat{\mathbf{x}}) - \delta_p(\mathbf{r}, \mathbf{x})) < 0. \quad (9)$$

We have

$$\delta_e(\mathbf{r}, \hat{\mathbf{x}}) - \delta_e(\mathbf{r}, \mathbf{x}) = 2 \sum_{i=1}^n e_i \nu_i + \sum_{i=1}^n e_i (e_i + 2b)$$

and

$$\delta_p(\mathbf{r}, \hat{\mathbf{x}}) - \delta_p(\mathbf{r}, \mathbf{x}) = 2 \sum_{i=1}^n (e_i - \bar{e}) \nu_i + \sum_{i=1}^n (e_i - \bar{e})^2,$$

where, for clerical convenience, we introduce the short-hand notation

$$e_i = x_i - \hat{x}_i \quad (10)$$

and

$$\bar{e} = \bar{x} - \bar{\hat{x}}. \quad (11)$$

Then, we may rewrite the above inequality (8) by

$$\zeta_0 + \sum_{i=1}^n \zeta_i \nu_i < 0, \quad (12)$$

where

$$\zeta_0 = \gamma \sum_{i=1}^n e_i (e_i + 2b) + (1 - \gamma) \sum_{i=1}^n (e_i - \bar{e})^2, \quad (13)$$

and

$$\zeta_i = 2\gamma e_i + 2(1 - \gamma)(e_i - \bar{e}), \quad 1 \leq i \leq n. \quad (14)$$

The noise samples ν_i are drawn from $N(0, \sigma)$, so that the probability that an error is made, equals

$$Pr(\delta(\mathbf{r}, \hat{\mathbf{x}}) < \delta(\mathbf{r}, \mathbf{x})) = Q\left(\frac{d(\mathbf{x}, \hat{\mathbf{x}})}{2\sigma}\right),$$

where the Q -function is defined by

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{u^2}{2}} du.$$

The ‘noise’ distance, $d(\mathbf{x}, \hat{\mathbf{x}})$, between the codewords \mathbf{x} and $\hat{\mathbf{x}}$ is given by

$$d(\mathbf{x}, \hat{\mathbf{x}}) = \frac{2\alpha}{\sqrt{\beta}}, \quad (15)$$

where

$$\alpha = \zeta_0 \quad (16)$$

and

$$\beta = \sum_{i=1}^n \zeta_i^2. \quad (17)$$

Working out yields

$$\begin{aligned} \alpha &= \gamma \sum_{i=1}^n e_i (e_i + 2b) + (1 - \gamma) \sum_{i=1}^n (e_i - \bar{e})^2 \\ &= -(1 - \gamma)n\bar{e}^2 + 2\gamma bn\bar{e} + \sum_{i=1}^n e_i^2 \end{aligned} \quad (18)$$

and

$$\begin{aligned} \beta &= 4 \sum_{i=1}^n [e_i - (1 - \gamma)\bar{e}]^2 \\ &= -4(1 - \gamma^2)n\bar{e}^2 + 4 \sum_{i=1}^n e_i^2. \end{aligned} \quad (19)$$

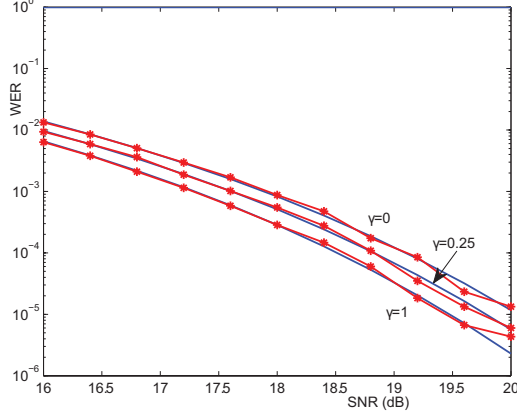


Fig. 1. Word error rate (WER) of the new detector as a function of the signal-to-noise ratio (SNR) for $q = 2$, $n = 8$ and $\gamma = 0$, $\gamma = 0.25$, and $\gamma = 1$. No offset, $b = 0$. The diagram shows the error performance computed by upperbound (20) and computer simulations (dotted lines).

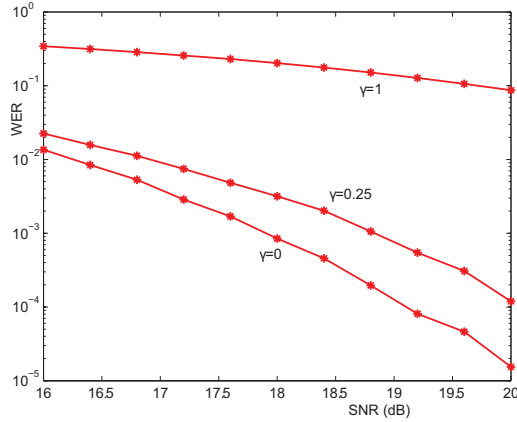


Fig. 2. Word error rate (WER) of the new detector as a function of the signal-to-noise ratio (SNR) for $q = 2$, $n = 8$ and $\gamma = 0$, $\gamma = 0.25$, and $\gamma = 1$. Offset $b = 0.3$. The diagram shows results by computer simulations. Note that the curve for $\gamma = 0$ (Pearson detection) is the same as shown in Figure 1.

The word error rate (WER) over all coded sequences \mathbf{x} is upperbounded by

$$\text{WER} < \frac{1}{|S_1|} \sum_{\mathbf{x} \in S_1} \sum_{\hat{\mathbf{x}} \neq \mathbf{x}} Q\left(\frac{d(\mathbf{x}, \hat{\mathbf{x}})}{2\sigma}\right). \quad (20)$$

Figure 1 shows results of computer simulations (dotted lines) and computations invoking (20) for $q = 2$, $n = 8$, and $\gamma = 0$, $\gamma = 0.25$, and $\gamma = 1$ for the matched case $b = 0$. The signal-to-noise-ratio (SNR), is defined by

$$\text{SNR}(\text{dB}) = -20 \log_{10} \sigma. \quad (21)$$

Figure 2 shows results of computer simulations (dotted lines) for the same parameters as used in Figure 1, but now with an offset $b = 0.3$. We notice that in this case the error

performance of the new detector improves upon that of the traditional Euclidean distance detector. At asymptotically high signal-to-noise ratios, the word error rate (WER) is overbounded by [8]

$$\text{WER} < \frac{2n(q-1)}{q} Q\left(\frac{d_{\min}}{2\sigma}\right), \quad (22)$$

where the minimum distance between any pair of codewords in S_1 , d_{\min} , is defined by

$$d_{\min} = \min_{\substack{\mathbf{x}, \hat{\mathbf{x}} \in S_1 \\ \mathbf{x} \neq \hat{\mathbf{x}}}} d(\mathbf{x}, \hat{\mathbf{x}}) = \min_{\substack{\mathbf{x}, \hat{\mathbf{x}} \in S_1 \\ \mathbf{x} \neq \hat{\mathbf{x}}}} \frac{2\alpha}{\sqrt{\beta}}. \quad (23)$$

We simply find

$$d_{\min} = \frac{1 - \frac{1-\gamma}{n} - 2\gamma|b|}{\sqrt{1 - \frac{1-\gamma^2}{n}}}. \quad (24)$$

Note that d_{\min} is independent of the alphabet size q . By varying the value of γ , we select an error performance that lies between that of a minimum Euclidean and a minimum Pearson distance detector. For the ideal case, $b = 0$, we find that the asymptotic *coding gain*, $\Gamma(\gamma)$, equals

$$\Gamma(\gamma) = \frac{1 - \frac{1-\gamma}{n}}{\sqrt{1 - \frac{1-\gamma^2}{n}}}. \quad (25)$$

The term $2\gamma|b|$ in (24) shows how the offset mismatch affects the error performance. The receiver will completely fail if

$$d_{\min} \leq 0$$

or

$$(b)_{\text{worst}}(\gamma) \geq \frac{1}{2\gamma} \left(1 - \frac{1-\gamma}{n}\right), \quad (26)$$

where $(b)_{\text{worst}}(\gamma)$ denotes the maximum offset that the receiver can tolerate just before collapsing. Clearly the maximum allowable offset $(b)_{\text{worst}}(\gamma)$ varies between 0.5 for Euclidean distance detection ($\gamma = 1$) and infinity for Pearson distance detection ($\gamma = 0$). Using (25) and (26), we plotted in Figure 3 the parametric equation of the maximum offset that the receiver can tolerate, $(b)_{\text{worst}}(\gamma)$, versus coding gain, $20 \log \Gamma(\gamma)$ (dB), for $n = 6, 8$ and 10 . The curves show the crucial trade-off between coding gain and offset tolerance. For example, for $n = 8$, the designer may buy for an investment of 0.4 dB coding loss, a maximum offset tolerance improvement by a factor of around six with respect to regular Euclidean distance detection. The receiver is completely immune to offset mismatch for the price of a coding loss of $10 \log(7/8) \approx 0.58$ dB.

IV. PARITY CHECK CODES

For certain applications it may be desirable to improve the noise margin at the expense of code redundancy. A parity check T -constrained code, denoted by S_p , consists

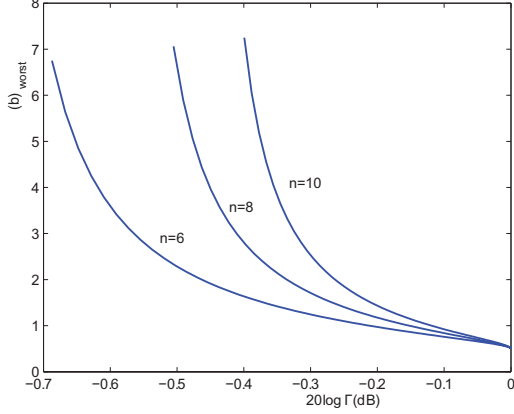


Fig. 3. Parametric equation of the maximum offset, $(b)_{\text{worst}}(\gamma)$, that the receiver can tolerate just before failure versus coding gain $20 \log \Gamma(\gamma)$ (dB) for $n = 6, 8$, and 10 .

of codewords, \mathbf{x} , taken from S_1 that additionally satisfy the parity check

$$\left(\sum_{i=1}^n x_i \right) \bmod q = a,$$

where $a \in \mathcal{Q}$ is an integer chosen by the code designer, for example, to maximize the code size.

Below we will concentrate on the binary case, $q = 2$, where for $a = (n + 1) \bmod 2$, the number of codewords equals 2^{n-1} , so that the rate of these codes is exactly $1 - 1/n$. We easily compute that the minimum distance, $d_{p,\min}$, between any pair of codewords equals

$$d_{p,\min} = \sqrt{2} \frac{1 - 2^{\frac{1-\gamma}{n}} - 2\gamma|b|}{\sqrt{1 - 2^{\frac{1-\gamma^2}{n}}}}. \quad (27)$$

The coding gain, $\Gamma_p(\gamma)$, under matched conditions, $b = 0$, equals

$$\Gamma_p(\gamma) = \sqrt{2} \frac{1 - 2^{\frac{1-\gamma}{n}}}{\sqrt{1 - 2^{\frac{1-\gamma^2}{n}}}}. \quad (28)$$

Figure 4 shows a comparison of the error performance of offset-resistant detection with and without extra parity check. For asymptotically large SNR, we may, for $n = 12$ and $\gamma = 0$, expect to gain a factor $\Gamma_p(0)/\Gamma(0) = \sqrt{20/11}$ (≈ 2.6 dB) in noise margin with respect to the non-parity case. We deduce from the diagram that the gain is around 1.9 dB in the SNR region shown.

The receiver will fail if

$$(b)_{p,\text{worst}}(\gamma) \geq \frac{1}{2\gamma} \left(1 - 2^{\frac{1-\gamma}{n}} \right). \quad (29)$$

We conclude that the parity check receiver is slightly more susceptible to offset mismatch than a standard receiver, see (26). We plotted in Figure 5 the parametric equation of the maximum offset that the receiver can tolerate, $(b)_{p,\text{worst}}(\gamma)$, versus coding gain, $20 \log \Gamma_p(\gamma)$, for $n = 6, 8$, and 10 .

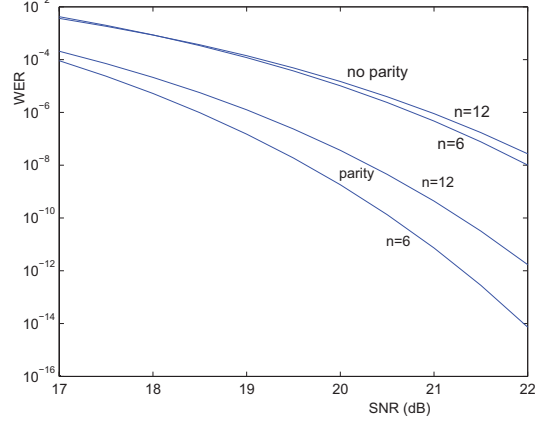


Fig. 4. Word error rate (WER) of offset-resistant detection, $\gamma = 0$, as a function of the signal-to-noise ratio (SNR) for $q = 2$, $n = 6$ and $q = 2$, $n = 12$, with and without parity check computed using upperbound (20).

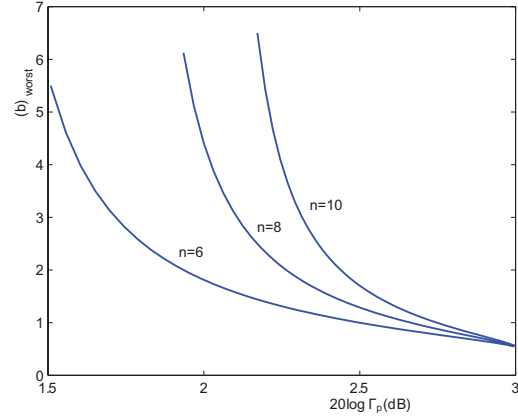


Fig. 5. Parametric equation of the maximum offset, $(b)_{p,\text{worst}}(\gamma)$, that the parity-check receiver can tolerate just before failure versus coding gain $20 \log \Gamma_p(\gamma)$ for $n = 6, 8$, and 10 .

V. IMPLEMENTATION ISSUES

The proposed distance measure (5) is a weighted sum of two distance measures, and, clearly, is more complicated to evaluate than a single distance. After combining (2), (3), (5), and (6), we may rewrite the distance as

$$\begin{aligned} \delta(\mathbf{r}, \hat{\mathbf{x}}) &= \sum_{i=1}^n r_i^2 - 2 \sum_{i=1}^n r_i \hat{x}_i + \sum_{i=1}^n \hat{x}_i^2 \\ &+ (1 - \gamma)(-n\bar{\hat{x}}^2 + 2\bar{\hat{x}} \sum_{i=1}^n r_i). \end{aligned} \quad (30)$$

The two right-hand terms are the extra load for the receiver with respect to traditional Euclidean detection, which is embodied by the three left-hand terms. Before we further discuss the complexity issue of the distance measure, we will first discuss how many evaluations of the distance are needed per received word. We tacitly assumed that the

receiver computes the distance (5) for all codewords in S_1 before it will give its verdict on the codeword sent. This would imply an exponential increase in the computational load with codeword length n . However, it was shown in [7] that the codebook S consists of K constant composition codes. A *constant composition code* is a set of codewords where the numbers of occurrences of the symbols within a codeword is the same for each codeword [9]. These specialize to *constant weight* codes in the binary case.

It was shown in [7] that we may significantly reduce the number of distance computations (5) to the number of constant composition codes, K , that constitute the full set S . For $q = 2$, since S_1 consists of $K = n$ constant weight codes, instead of 2^n for a full set only n computations are required. For a binary parity-check code the number of constant weight subsets equals $(n+1)/2$ for n odd and $n/2$ for n even, which reduces the number of computations even more.

Let a constant composition code be denoted by S_{w_i} , $1 \leq i \leq K$. Each constant composition code, S_{w_i} , is characterized by one codeword of the code, called *pivot word*, denoted by x_{p_i} , $1 \leq i \leq K$. The pivot word x_{p_i} is, by definition, the largest word in the lexicographical ordering (the largest symbols first) of the codewords in S_{w_i} . Thus, for a pivot word we have $x_{p_i} \in S_{w_i}$ and $x_{p_i} = (x_{p_i,1}, x_{p_i,2}, \dots, x_{p_i,n})$, where $x_{p_i,1} \geq x_{p_i,2} \geq x_{p_i,3} \dots \geq x_{p_i,n-1} \geq x_{p_i,n}$, $1 \leq i \leq K$. The remaining codewords in S consist of all distinct sequences that can be formed by permuting the order of the n symbols of each pivot word x_{p_i} , $1 \leq i \leq K$.

Detection of the received vector, r , is accomplished by invoking the following two-step procedure:

- 1) The n symbols of the received word, r , are sorted, largest to smallest, in the same way as taught in Slepian's prior art [10].
- 2) Compute the distance, using (30), between the sorted received word and all K pivot words. The receiver decides that a word taken from S_{w_k} , $1 \leq k \leq K$, was sent in case the pivot word x_{p_k} is at minimum distance to the received vector r . Now that we know that the word sent is a member of S_{w_k} , we can decode the received word by Slepian's procedure for single permutation codes.

In an embodiment of the second part of the above procedure, using a micro-processor, for example, the receiver has tabulated and stored in memory the K pivot words. In addition, the receiver has, for each pivot word, pre-computed the terms, see (3),

$$A_j = \frac{1}{n} \sum_{i=1}^n x_{p_j,i}, \quad 1 \leq j \leq K$$

and, see (30),

$$B_j = \sum_{i=1}^n x_{p_j,i}^2 - n(1-\gamma)A_j^2, \quad 1 \leq j \leq K,$$

which are independent of r . The receiver computes for each received codeword

$$C = 2 \sum_{i=1}^n r_i$$

and for each of the K pivot words, see (30),

$$D_j = -2 \langle \tilde{r}, x_{p_j} \rangle + B_j + CA_j, \quad 1 \leq j \leq K,$$

where $\langle u, v \rangle$ denotes the inner product of the vectors u and v , and \tilde{r} denotes the sorted received vector. Note that we do not need to compute the term $\sum r_i^2$, since it is irrelevant. As we may observe from the above, the main computational load is the computation of the inner product. This is also the case for Euclidean detection, and we conclude that the hybrid detection is only slightly more involved than Euclidean distance detection.

VI. CONCLUSIONS

We have presented a hybrid detection method, where the distance measure is a weighted sum of the Euclidean and Pearson distance, which offers the designer the option of trading noise margin versus amount of immunity against unknown offset. We have computed the error performance of the hybrid detection method in the presence of unknown offset and additive noise. We have computed the maximum allowable offset, $(b)_{\text{worst}}$, and the penalty in noise margin. We have introduced parity-check T -constrained codes, and have shown that with a code rate $1 - 1/n$, we may gain around 2-3 dB in noise margin with respect to regular T -constrained codes. We have discussed issues related to the computational load of the hybrid detector.

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